UNSOLVED PROBLEMS

Edited by: **Richard Guy & Richard Nowakowski**

In this department the MONTHLY presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial or related results. Typescripts should be sent to Richard Guy, Department of Mathematics & Statistics, The University of Calgary, Alberta, Canada T2N 1N4.

Coin-Weighing Problems

Richard K. Guy and Richard J. Nowakowski

The question of finding a single counterfeit coin from a set of regular coins in the fewest number of weighings using just a balance beam has been a notorious problem. The regular coins are all the same weight while the counterfeit coin is a different weight.

The problem was popular on both sides of the Atlantic during World War II ([**14**, **15**, **20**, **21**, **27**, **34**, **39**, **46**]; it was even suggested that it should be dropped over Germany in an attempt to sabotage their war effort; see [**35**, **40**, **43**] for some history. In solving [**39**] Kaplansky, Neugebauer and Pennell gave the following general solution to the problem of underweight counterfeit coins: *If* $3^{n-1} \leq N < 3^n$ *then n weighings suffice to show if there is (and to identify) a counterfeit coin among N coins*. If it is known that a counterfeit coin exists then n weighings will identify the coin from among N coins if $3^{n-1} < N \leq 3^n$. Dyson [**12**] gave an elegant solution using ternary labels when it is not known if the counterfeit coin is heavy or light; see [**11**] for a solution in verse. In this case, n weighings suffice

(i) if $N \leq (3^n - 3)/2$ and it is required to find if the dud is heavy or light;

(ii) if $N \leq (3^n - 1)/2$, given an extra coin known to be good, and it is required to find if the dud is heavy or light; and

(iii) $N \leq (3^n + 1)/2$ if there is a good coin but the relative weight of the counterfeit coin is not required.

In the solution to the general problem posed in [**15**], the editors note that all the solutions so far consider the coins to be distinguishable when in the balance pan. They show that if the coins in a scale pan are to be considered as a single set, then n weighings will find a coin amongst $N \leq (7 \times 3^{n-2} - 1)/2$.

There are many other variants [**1**, **7**, **9**, **32**]. Forysthe, a responder to [**14**], seems to be the first to ask the question using a spring balance i.e. a weighing device that will return the exact weight; see also [**33**, **41**]. Christen [**8**] asks the question for two counterfeit coins but of complementary weights. Hwang [**25**] proposes and analyses many weighing schemes.

Shapiro's problem [**41**] assumes N coins, $N - 1$ of weight a and one of weight b where a and b are known, and, as with Forsythe, an accurate scale. He asks for the least number of weighings to determine which coin has weight b, where the weighing scheme must be given in advance. Söderberg and Shapiro [**44**] ask the more general question of how many weighings are needed to determine which of N coins are of weight a and which of weight b if the numbers of each are not known. Denote this number of weighings by $f(N)$ then they show that (i) $f(N) \geq N/\log_2(N+1)$; (ii) $f(3^{m-1}(3+m)) \leq 3^m$; (iii) $f(5^{m-1}(2m+5)) \leq 5^m$; and that (iv) $f(N) = O(N/\ln N)$. In addition Erdös and Rényi [**13**] (and several others independently) show that

$$f(N) = \frac{N}{\log_4 N} + O\left(\frac{N \ln \ln N}{(\ln N)^2}\right)$$

Cantor and Mills [**6**] and Lindström [**28**, **29**, **30**] give explicit weighing schemes for $N = 2^{k-1}k$ (also see [**1**]). The result of Liu [**31**] is not as good as this.

Another variant is that of deciding which coins are counterfeit out of N coins but the number of weighings is fixed. The "Lower Slobbovian Counterfeiters" [**17**, **4**, **24**] and ApSimon's Mints problem [**2**, **23**] are examples.

Some years ago Sir Alexander Oppenheim reminded us of yet another variant. It was perhaps first stated by Bellman and Gluss [**3**]: use a beam balance to find k counterfeit coins among N coins where all the counterfeit coins are of the same weight. Let $w(N,k)$ be the least number of weighings required to find the k lighter coins. It is easy to see that $w(N,k)$ is at least $\log_3\binom{N}{k}$. Pyber [**36**] showed that if there were no more than m light coins then they could be identified in $\left\lceil \log_3\binom{n}{m} \right\rceil + 15m$ weighings.

The case $k = 2$ has recently attracted attention [**5**, **25**, **45**]. Tošić gave weighing procedures which improved on those of Cairns and showed that the lower bound could be attained apart from one possible extra weighing. For example, with seven coins weigh 123 against 456. If they balance, weigh 1 against 2 and 4 against 5. If 123 are heavier, then 4567 contain two light coins which can be determined in two more weighings. In the special case $w(N = 3^m, 2) = 2m$, the extra weighing is never needed.

In which cases is $w(N, 2) = \lceil \log_3\binom{N}{2} \rceil + 1$? Is $N = 13$ the first?

If there are three lighter coins, then there is no new problem until we get to $w(6,3) = 3$ which was the subject of a problem in [**10**]. Oppenheim showed that $w(7,3) = w(8,3) = 4$: first weigh three coins against three. Nine coins require 5 weighings. Can the lighter coins be identified in five weighings if $N = 12$? 3^m coins can be sorted in $3m$ weighings; this can be improved by at most one weighing; when?

If $k = 4$ we know that $w(8,4) = w(9,4) = 5$; although $\binom{8}{4} = 70 < 3^4$ it is not possible to make a weighing among 8 coins with 4 light each of whose outcomes leave less than 26 possibilities and while $26 < 3^3$ they cannot be separated by three weighings. We can show that $w(3^m, 4) \leq 4m - 1$.

If $k = 5$, then $w(10,5) = 6$ and we can show that $w(3^m, 5) \leq 5m$, but this can almost certainly be improved.

The problem of Söderberg and Shapiro, but using a beam balance in place of a spring balance is: given N coins which each weigh one of two weights, determine the least number, $W(N)$, of weighings required to find which coins are of each weight. Of course, if all the coins are of the same weight, we won't be able to say which of the two possible weights they are. We see that $W(1) = 0$, $W(2) = 1$, $W(3) = 2$, $W(4) = 3$, $W(5) = 4$, $W(6) = 4$ and generally $W(N) \geq \lceil \log_3 2^N \rceil$. For which N is there equality?

Notice that we don't require that the whole weighing scheme be given in advance, as has been done in the more elegant solutions of the famous 12 coin problem. The subsequent weighings depend on the results of the previous ones. We could also ask for the minimum number of weighings, if these are all to be prescribed before the first weighing is made.

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Department of Mathematics
The University of Calgary
Calgary, Alberta
CANADA T2N 1N4

Department of Mathematics
Dalhousie University
Halifax, Nova Scotia
CANADA B3H 3J5

Bridges would not be safer if only people who knew the proper definition of a real number were allowed to design them.

—*Norman David Mermin* (1935–)
Topological Theory of Defects in *Review of Modern Physics*
July 1979 51, No. 3.